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GAUGE PRINCIPLE AND VARIATIONAL FORMULATION FOR FLOWS OF AN IDEAL FLUID

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ABSTRACT: A gauge principle is applied to mass flows of an ideal compressible fluid subject to Galilei transformation. A free-field Lagrangian defined at the outset is invariant with respect to global $SO(3)$ gauge transformations as well as Galilei transformations. The action principle leads to the equation of potential flows under constraint of a continuity equation. However, the irrotational flow is not invariant with respect to local $SO(3)$ gauge transformations. According to the gauge principle, a gauge-covariant derivative is defined by introducing a new gauge field. Galilei invariance of the derivative requires the gauge field to coincide with the vorticity, i.e. the curl of the velocity field. A full gauge-covariant variational formulation is proposed on the basis of the Hamilton's principle and an assoicated Lagrangian. By means of an isentropic material variation taking into account individual particle motion, the Euler's equation of motion is derived for isentropic flows by using the covariant derivative. Noether's law associated with global $SO(3)$ gauge invariance leads to the conservation of total angular momentum. In addition, the Lagrangian has a local symmetry of particle permutation which results in local conservation law equivalent to the vorticity equation.

KEY WORDS: ideal fluid flow, gauge theory, $SO(3)$, vorticity, variational formulation

1 INTRODUCTION

It is generally accepted that investigation of vorticity dynamics is essential for full understanding of fluid motions. It will be found in the present paper that the vorticity field and its dynamics are closely related with various symmetries of fluid flows. Fluid mechanics is considered a field theory of fluid flows in Newtonian mechanics, in other words, a field theory of mass flow subject to Galilei transformation.

There are various similarities between fluid dynamics and electromagnetic phenomena. For example, the functional relation between the velocity and vorticity field is the same as the Biot-Savart law in electromagnetism between the magnetic field and electric current[1]. Sound scattering by vortices[2~4] is analogous to electron scattering by magnetic field. Scattering of shallow water waves is investigated in Refs.[5,6] as an analogy with the Aharonov-Bohm effect in quantum mechanics[7]. Furthermore, there is a law which may be called acoustic Faraday's law*[8~10]. Thus, one might ask whether the similarities are mere an analogy, or have a solid theoretical background.

In the theory of gauge field, a guiding principle is that laws of physics should be expressed in a form that is independent of any particular coordinate system**. To begin with in Section 2, we review the scenario of the gauge theory in the quantum field theory and particle physics[11,12]. A free-particle Lagrangian is defined first in such a way as having an invariant form under the Lorenz transformation. Next, a gauge principle is applied to the Lagrangian, requiring it to have a symmetry, namely the gauge invariance. Thus, a

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* This is named in [10] to mean a phenomenon in which an acoustic wave is generated by a vortex ring moving near a soild body and its signal depends on the rate of change of flux (through the vortex ring) of an imaginary potential flow around the body.
** The Lorentz transformation can be dervied by requiring that the wave equation is invariant between two frames relative motion[20].

gauge field such as the elecromagnetic field is introduced to satisfy the local gauge invariance.

There are obvious differences between the fluid-flow field and the quantum field. Firstly, the field of fluid flow is non-quantum, which however causes no problem since the gauge principle is independent of the quantization principle. In addition, the fluid flow is subject to the Galilei transformation instead of Lorenz transformation. This is not an obstacle because the former is a limiting transformation of the latter as the ratio of a representative velocity to the light speed tends to zero. Thirdly, relevant gauge groups should be different. This is resolved in terms of a concept of isometric gauge transformation. In Section 3, a gauge transformation by a rotation group $SO(3)$ is considered as a group relevant to fluid flows.

Gauge theory of rotation invariant Lagrangian with an internal $O(3)$-symmetry was studied already for the Bohr model of nuclear collective rotation of a finite number of modes[13]. It will be found below that there are various similarities between this system (of five field variables) and the present problem of fluid flows (of infinite dimensions). A similarity especially to be noted is the fact that both systems are considered to be dynamical systems, in other words, so-called $d = 1$ field theory in the sense that the gauge field is defined only for the covariant derivative of time evolution. The gauge field of this model was found to be the angular velocity .

In the present paper for fluid flows, we seek a scenario which has a formal equivalence with the gauge theory in the particle physics. In order to go further over a mere analogy of the flow field to the gauge field, in Section 4, we define a Galilei-invariant free-field Lagrangian for fluid flows and examine whether it has global $SO(3)$-gauge invariance in addition to the Galilei invariance. It will be shown that the velocity field obtained by the variational principle is irrotational, i.e. a potential flow.

In Section 5, the potential flow will be shown to be not invariant to local gauge transformations, although it is invariant with respect to global gauge transformation and Galilei transformation. It will be shown that the local transformation introduces a new rotational component in the velocity field (i.e. vorticity), even though the original field does not have the vorticity. In complying with the local gauge invariance, a gauge-covariant derivative is defined by introducing a new gauge field. Galilei invariance of the covariant derivative requires that the gauge field should coincide with the vorticity, which is twice the angular velocity of local fluid rotation. As a result, the covariant derivative of velocity is found to be the so-called material time derivative of velocity. This finding is reported in Ref.[14] as a brief communication paper.

Present analysis suggests a new variational formulation. So far, there are known two approaches in the variational formulations[15~19]. One is the variation with respect to Lagrangian particle coordinates and the other is the one with respect to field variables (Eulerian field variables). The variational formulation of the Eulerian fields must be supplemented with additional conditions of mass conservation and entropy conservation, and the Lin's constraint as well in order to incorporate a particle aspect in the variational formulation. In Section 6, using the notion of isentropic material variation together with the gauge-covariant derivative, the Euler's equation of motion for rotational isentropic flows is derived from the variational principle equivalent to the Hamilton's principle. This formulation is considered to fill in the gap between above two approaches, i.e. Lagrangian approach and Eulerian one. The isentropic material variations satisfy the conservation conditions of mass and entropy within itself, while the Lagrangian functional is represented by field variables with no additional constraint.

Regarding the material variation without using additional constraints, there is a similarity to the Bretherton's formulation[18]. However, an important difference should be remarked. Namely, in his formulation, the expression of the material time derivative of velocity is given in advance, or taken as an identity, while in the present formulation the covariant derivative (which is equivalent to the material derivative of velocity) is derived from the gauge principle. In fact, this is a central point of the present paper.

Another remarkable point in the present approach is that Eulerian variation of the Lagrangian defined at the outset results in an equation for a potential flow of a homentropic fluid, which is taken as a free-field of fluid flow. The Lagrangian functional does not include a direct description of individual particle motion which connects the particle position with the velocity field, except the continuity equation. A complete variational approach taking into account a rotational field should be carried out with a material variation in terms of the covariant derivative, which takes into account individual particle motion.

There are some byproducts from the present formulation. The global $SO(3)$ gauge invariance implies a Noether's conservation law, which is found to be the conservation of total angular momentum. In addition, the Lagrangian has a symmetry with respect to particle permutation, which leads to a local law of vorticity conservation, i.e. the vorticity equation. Thus, it is found that the well-known equations are related to ceratin symmetries of the fluid system.

2 CONCEPTUAL SCENARIO OF THE GAUGE PRINCIPLE

Typical successful examples of the gauge theory are Dirac equation or Yang-Mills equation in particle physics. Free-particle Lagrangian function Λ_{free}, e.g. for a free electron, is constructed so as to be invariant under the Lorenz transformation of space-time (x^μ), where

$$\Lambda_{\text{free}} = \bar{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$
$$\bar{\psi} = \psi^\dagger\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{1}$$

m is the mass, ψ is a Dirac wave function of four components for electron and positron with $\pm 1/2$ spins, $\psi^\dagger$ the hermitian conjugate of ψ, and γ^μ the Dirac matrices with $\mu = 0, 1, 2, 3$ and $x^0 = t$ (time), I being 2×2 unit matrix. In the Yang-Mills case, the wave functions are considered to represent internal states such as up-down quarks.

The above Lagrangian has a symmetry called a global gauge invariance. Namely, its form is invariant under the transformation of the wave function, e.g. $\psi \mapsto \mathrm{e}^{\mathrm{i}\alpha}\psi$ for an electron field. The term "global" means that the phase α is a real constant, i.e. independent of coordinates. This keeps the probability density, $|\psi|^2$, unchanged*.

In addition, we should be able to have invariance under a local gauge transformation,

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x) = g(x)\psi(x) \tag{2}$$

where $\alpha = \alpha(x)$ varies with the space-time coordinates $x = (x^\mu)$. With this transformation too, the probability density $|\psi|^2$ obviously is not changed.

However, the free-particle Lagrangian Λ_{free} is not invariant under such a transformation because of the derivative opearator $\partial_\mu = \partial/\partial x^\mu$ in Λ_{free}. This demands that some background field interacting with the particle should be taken into account: Electromagnetic field or Gauge field. If a new gauge field term is included in the Lagrangian function Λ, then the local gauge invariance will be attained[11,12]. This is the Weyl's principle of gauge invariance.

If a proposed Lagrangian including a partial derivative of some matter field ψ is invariant under global gauge transformation as well as Lorentz transformation, but not invariant under local gauge transformation, then the Lagrangian is to be altered by replacing the partial derivative with a covariant derivative including a gauge field $A(x)$ (compatible with the gauge transformation), $\partial \to \nabla = \partial + A(x)$, so that the Lagrangian function Λ acquires local gauge invariance. The second term $A(x)$ is called also connection. The point of introducing the gauge field is to obtain a generalization of the gradient that transforms as

$$\begin{aligned}\nabla\psi \mapsto \nabla'\psi' &= (\partial + A')g(x)\psi \\ &= g(x)(\partial + A)\psi = g(x)\nabla\psi\end{aligned} \tag{3}$$

where $\psi' = g(x)\psi$. In dynamical systems which evolve with respect to the time coordinate t, the replacement $\partial \to \nabla = \partial + A(x)$ is made only for the t derivative. As is done in the $d = 1$ field theory[13], the gauge field in this case is identified as the angular velocity.

Finally, the principle of least action is applied

$$\delta\mathcal{A} = 0 \qquad \mathcal{A} = \int_{t_0}^{t_1} \Lambda(\psi, A)\mathrm{d}t \tag{4}$$

with fixed end conditions at t_0 and t_1, where $\mathcal{A}$ is the action function.

(1) In the case of the wave function ψ representing an electron field, the local gauge transformation (2) of ψ is represented by means of an element $g(x) = \mathrm{e}^{\mathrm{i}q\alpha(x)} \in U(1)$** at each point x (q : a real constant), written as $\psi'(x) = \mathrm{e}^{\mathrm{i}q\alpha(x)}\psi(x)$. The gauge-covariant derivative is then defined by

$$\nabla_\mu = \partial_\mu - \mathrm{i}qA_\mu(x) \tag{5}$$

where q is a constant***, and $A_\mu = (-\varphi, A_k)$ is the electromagnetic potential (four-vector potential with the electric potential φ and magnetic three-vector potential A_k, $k = 1, 2, 3$). The electromagnetic potential (connection term) transforms as

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x) \tag{6}$$

It is not difficult to see that this satisfies the relation (3), $\nabla'\psi' = g\nabla\psi$. Thus, the Dirac equation with an electromagnetic field is derived. (See Appendix A for its brief summary.)

* The global gauge invariance results in conservation of Noether current[11,12]. See §6.4.

** The unitary group $U(1)$ is the group of complex numbers $z = \mathrm{e}^{\mathrm{i}\theta}$ of absolute value 1.

*** $q = e/(\hbar c)$ with c the light-speed, e the electric-charge, and $\hbar$ the Planck constant.

(2) In the second example of Yang-Mills's formulation of up-down quark field, the local gauge transformation of the form (2) is represented by $\boldsymbol{g}(x) \in SU(2)^*$. Consider an infinitesimal gauge transformation represented by

$$\begin{aligned}\boldsymbol{g}(x) &= \exp\left[\mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}(x)\right] \\ &= \boldsymbol{I} + \mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}(x) + O(|\boldsymbol{\alpha}|^2) \qquad |\boldsymbol{\alpha}| \ll 1 \end{aligned} \tag{7}$$

where $\boldsymbol{\sigma}\cdot\boldsymbol{\alpha} = \boldsymbol{\sigma}_1\alpha^1 + \boldsymbol{\sigma}_2\alpha^2 + \boldsymbol{\sigma}_3\alpha^3$ with real functions $\alpha^k(x)(k = 1, 2, 3)$, and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ are the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$$

$$\boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

composing a basis of the Lie algebra $SU(2)$ which is considered as a real 3-dimensional vector space**. The commutation relations are given by***

$$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\mathrm{i}\epsilon_{jkl}\boldsymbol{\sigma}_l \tag{8}$$

The second term on the right hand side of Eq.(7) is a generator of an infinitesimal gauge transformation. The gauge-covariant derivative is represented by

$$\nabla_\mu = \partial_\mu - \mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu(x) \tag{9}$$

where the connection consists of three terms, $\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu = \boldsymbol{\sigma}_1 A^1_\mu + \boldsymbol{\sigma}_2 A^2_\mu + \boldsymbol{\sigma}_3 A^3_\mu$ in accordance with the 3-dimensionality of $SU(2)$, and q the coupling constant of interaction. The connection $\boldsymbol{A}_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$ transforms as

$$\boldsymbol{A}'_\mu = \boldsymbol{A}_\mu - 2q\boldsymbol{\alpha}\times\boldsymbol{A}_\mu + \partial_\mu\boldsymbol{\alpha} \tag{10}$$

instead of Eq.(6). The following three gauge fields (colors) are thus introduced: $\boldsymbol{A}^k = (A^k_0, A^k_1, A^k_2, A^k_3)$ with $k = 1, 2, 3$, which are the Yang-Mills gauge fields. One characteristic feature distinct from the previous electrodynamic case is the non-abelian nature of the algebra $SU(2)$, represented by the second term of Eq.(10) resulting from the non-abelian commutation rule (8). It is interesting to find that the transformation law in the $d = 1$ field theory (Eq.(3.4b) of Ref.[13]) is equivalent to the non-abelian law (10) if $2\boldsymbol{A}_\mu$ is replaced by the angular velocity $\boldsymbol{\omega}$ and $2\boldsymbol{\alpha}$ by an infinitesimal rotation angle $\delta\boldsymbol{\theta}$.

In the subsequent sections, we consider fluid flows and try to formulate the flow field on the basis of the gauge principle. In this case, the group of gauge transformation is the rotation group $SO(3)$, which is also non-abelian. It would not be surprising if we obtain the same transformation law as Eq.(10).

3 PRELIMINARY STUDY OF VELOCITY FIELD

3.1 Gauge Transformation of Velocity Field

As a preliminary analysis, we consider the total kinetic energy $\mathcal{K}$ of fluid-flow,

$$\mathcal{K}[\boldsymbol{v}] = \frac{1}{2}\int_M \langle\boldsymbol{v}(x), \boldsymbol{v}(x)\rangle\ \rho(x)\mathrm{d}^3x$$

$$\langle\boldsymbol{v}, \boldsymbol{v}\rangle = (v^1)^2 + (v^2)^2 + (v^3)^2$$

where $\boldsymbol{v} = (v^1, v^2, v^3)$ is the fluid velocity, and $\rho(x)$ the fluid density, M being the space of fluid flow under consideration.

With a fixed element of the group $SO(3)$, $\boldsymbol{R} \in SO(3)^{****}$, a rotational transformation of a tangent vector $\boldsymbol{v}$ (velocity vector) is represented by $\boldsymbol{v}' = \boldsymbol{R}\boldsymbol{v}$. With this transformation, the magnitude $|\boldsymbol{v}|$ is invariant, that is isometric: $\langle\boldsymbol{v}', \boldsymbol{v}'\rangle = \langle\boldsymbol{R}\boldsymbol{v}, \boldsymbol{R}\boldsymbol{v}\rangle = \langle\boldsymbol{v}, \boldsymbol{v}\rangle$, which is not more than the definition of the orthogonal transformation. In matrix notation $\boldsymbol{R} = (R^i_j)$, a vector v^i is mapped to $(v')^i = R^i_j v^j$. Hence

$$\begin{aligned}\langle\boldsymbol{v}', \boldsymbol{v}'\rangle &= (v')^i(v')^i = R^i_j v^j\ R^i_k v^k \\ &= v^k v^k = \langle\boldsymbol{v}, \boldsymbol{v}\rangle\end{aligned}$$

where the orthogonal transformation is described by

$$R^i_j R^i_k = (R^{\mathrm{T}})^j_i R^i_k = (R^{\mathrm{T}}R)^j_k = \delta^j_k \tag{11}$$

and $\boldsymbol{R}^{\mathrm{T}}$ is the transpose of $\boldsymbol{R}$, equal to the inverse $\boldsymbol{R}^{-1}$. Using the unit matrix $\boldsymbol{I} = (\delta^j_k)$, this is rewritten as

$$\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I} \tag{12}$$

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* $SU(2)$ is the special unitary group, consisting of complex 2×2 matrices $g = (g_{ij})$ with $\det g = 1$. The hermitian conjugate $g^\dagger = (g^\dagger_{ij}) = (\bar{g}_{ji})$ is equal to g^{-1} where the overbar denotes complex conjugate. Its Lie algebla $SU(2)$ consists of skew hermitian matrices of trace 0.

** The vector space $SU(2)$ is closed under multiplication by real numbers α^k (e.g. see [11]).

*** The structure constant ϵ_{jkl} takes 1 or -1 according as (jkl) is an even or odd permutation of (123), and 0 if (jkl) is not a permutation of (123).

**** $SO(3)$ is a group of special orthogonal transformations of $\mathcal{R}^3$ characterized with $\det\boldsymbol{R} = 1$.

The total kinetic energy $\mathcal{K}$ is also invariant, since the mass in a volume element d^3x is invariant by the rotational transformation, $\boldsymbol{R}[\rho\mathrm{d}^3x] = \rho'\boldsymbol{R}[\mathrm{d}^3x] = \rho\mathrm{d}^3x$, where $\rho' = \rho$ since scalars are invariant by the transformation. Thus, $\mathcal{K}$ has the global gauge invariance.

Similarly, it is not difficult to see that the kinetic energy $\mathcal{K}$ is invariant under a local transformation, $\boldsymbol{v}(x) \mapsto \boldsymbol{v}'(x) = \boldsymbol{R}(x)\boldsymbol{v}(x)$ depending on a point x, where $\boldsymbol{R}(x) \in so(3)$ at $\forall x \in M$

$$\int \langle \boldsymbol{R}(x)\boldsymbol{v}(x), \boldsymbol{R}(x)\boldsymbol{v}(x)\rangle \boldsymbol{R}(x)[\rho(x)\mathrm{d}^3x] = \int \langle \boldsymbol{v}(x), \boldsymbol{v}(x)\rangle \rho(x)\mathrm{d}^3x$$

Thus, the local gauge invariance is also satisfied with the kinetic energy $\mathcal{K}$.

3.2 Galilei Transformation of a Velocity Field

The quantum field is subject to the Lorenz transformation, whereas the field of fluid flow is subject to the Galilei transformation. This difference is not an essential obstacle to the formulation, because the Galilei transformation is regarded as a limiting transformation of the Lorentz transformation of space-time $(x^\mu) = (ct, \boldsymbol{x})$ as $v/c \to 0$*. In the Lorentz transformation, a line-element of world-line is a vector, $\mathrm{d}\boldsymbol{s} = (c\mathrm{d}t, \mathrm{d}\boldsymbol{x})$ with the Minkowski metric $g_{ij} = \mathrm{diag}(-1, 1, 1, 1)$, and its length $|\mathrm{d}\boldsymbol{s}|$ is a scalar, namely a Lorentz-invariant, where

$$\begin{aligned} |\mathrm{d}\boldsymbol{s}|^2 &= -c^2\mathrm{d}t^2 + \langle \mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}\rangle \\ &= -c^2(\mathrm{d}t')^2 + \langle \mathrm{d}\boldsymbol{x}', \mathrm{d}\boldsymbol{x}'\rangle \end{aligned} \tag{13}$$

between two frames, $(t, \boldsymbol{x})$ and $(t', \boldsymbol{x}')$, and the light speed c is an invariant[20].

Scalar product of a 4-momentum $P = (E/c, \boldsymbol{p})$ of a particle of mass m with the line element $\mathrm{d}\boldsymbol{s}$ is

$$\begin{aligned} (P, \mathrm{d}\boldsymbol{s}) &= -\frac{E}{c}c\mathrm{d}t + \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{x} = (\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - E)\mathrm{d}t \\ &= m(\boldsymbol{v}^2 - c^2)\mathrm{d}t = -m_0c^2\mathrm{d}\tau \end{aligned} \tag{14}$$

where m_0 is the rest mass, $\boldsymbol{v}$ and $\boldsymbol{p} = m\boldsymbol{v}$ are 3-velocity and 3-momentum of the particle, respectively[11,21], and

$$E = mc^2 \qquad m = \frac{m_0}{(1-\beta^2)^{1/2}}$$
$$\beta = \frac{v}{c} \qquad \mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$$
$$\mathrm{d}\tau = (1-\beta^2)^{1/2}\mathrm{d}t \quad \text{(proper time)}$$

* Spatial components are denoted by bold letters.

The leftmost side of Eq.(14) is a scalar product, i.e. an invariant with respect to the Lorentz transformation, and the rightmost side $-m_0c^2\mathrm{d}\tau$ is an invariant as well, since Eq.(13) gives $|\mathrm{d}\boldsymbol{s}|^2 = -c^2\mathrm{d}t^2(1-\beta^2) = -c^2\mathrm{d}\tau^2$. The function $\boldsymbol{\Lambda} = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - E$ is what is called the Lagrangian in Mechanics. Hence it is found that either of the five expressions of Eq.(14) might be taken as the integrand of the action $\boldsymbol{\mathcal{A}}$ of Eq.(4).

Next, we consider a Lorenz-invariant Lagrangian $\Lambda_L^{(0)}$ in the limit as $v/c \to 0$, and seek its appropriate counterpart Λ_G in the Galilei system. In this limit, the mass m and energy mc^2 are approximated by m_0 and $m_0(c^2 + v^2/2 + \epsilon)$ respectively in a macroscopic fluid system (see Ref.[22], §133), where ϵ is the internal energy per unit fluid mass. The first expression of the second line of Eq.(14) is, then asymptotically

$$\begin{aligned} mv^2 - mc^2 &\Rightarrow m_0v^2 - m_0\Big(c^2 + \frac{1}{2}v^2 + \epsilon\Big) \\ &= (\rho\mathrm{d}^3x)\Big(\frac{1}{2}v^2 - \epsilon - c^2\Big) \end{aligned}$$

where m_0 is replaced by $\rho(x)\mathrm{d}^3\xi$. Thus, the Lagrangian $\Lambda_L^{(0)}$ would be defined by

$$\Lambda_L^{(0)}\mathrm{d}t = \int_M \mathrm{d}^3x\rho(x)\Big(\frac{1}{2}\langle \boldsymbol{v}(x), \boldsymbol{v}(x)\rangle - \epsilon - c^2\Big)\mathrm{d}t \tag{15}$$

The third $-c^2\mathrm{d}t$ term is necessary so as to satisfy the Lorenz-invariance (see Ref.[21], §87). It is obvious that the term $\langle \boldsymbol{v}(x), \boldsymbol{v}(x)\rangle$ is not invariant with the Galilei transformation, $\boldsymbol{v} \mapsto \boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{U}$. Using the relations $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$ and $\mathrm{d}\boldsymbol{x}' = \boldsymbol{v}'\mathrm{d}t = (\boldsymbol{v} - \boldsymbol{U})\mathrm{d}t'$ with respect to two frames of reference moving with a relative velocity $\boldsymbol{U}$, the invariance (13) leads to

$$\mathrm{d}t' = \mathrm{d}t + \Big(\frac{1}{c^2}\Big(-\langle \boldsymbol{v}, \boldsymbol{U}\rangle + \frac{1}{2}\boldsymbol{U}^2\Big) + O((v/c)^4)\Big)\mathrm{d}t \tag{16}$$

The second term of $O((v/c)^2)$ on the right hand side makes the Lagrangian $\Lambda_L^{(0)}\mathrm{d}t$ Lorenz-invariant exactly in the $O((v/c)^0)$ terms in the limit as $v/c \to 0$.

When we consider a fluid flow as a Galilei system, the following prescription is applied. Suppose that the flow is investigated in a finite domain M in space. Then the $c^2\mathrm{d}t$ term gives a constant $c^2\boldsymbol{\mathcal{M}}\mathrm{d}t$ to $\Lambda_L^{(0)}\mathrm{d}t$, where $\boldsymbol{\mathcal{M}} = \int_M \mathrm{d}^3x\rho(x)$ is the total mass in the domain M. In carrying out variation of $\boldsymbol{\mathcal{A}}$, the total mass $\boldsymbol{\mathcal{M}}$ is fixed as a constant. Next, keeping this in mind implicitly, we define the Lagrangian Λ_G of a fluid motion in the Galilei system by

$$\Lambda_G \mathrm{d}t = \int_M \mathrm{d}^3 x \rho(x) \Big(\frac{1}{2}\langle \boldsymbol{v}(x), \boldsymbol{v}(x)\rangle - \epsilon\Big)\mathrm{d}t \qquad (17)$$

Only when we need to consider its Galilei invariance, we use the Lagrangian $\Lambda_L^{(0)}$. In the Lagrangian formulation of subsequent sections, local conservation of mass is imposed. As a conequence, the mass is conserved globally. Thus, the use of Λ_G will not cause serious problem except the case requiring its Galilei invariance*.

Under the Galilei transformation from one frame x to another x' moving with a relative velocity $\boldsymbol{U}$, we have

$$\begin{aligned} &x = (t, \boldsymbol{x}) \Rightarrow x' = (t', \boldsymbol{x}') = (t, \boldsymbol{x} - \boldsymbol{U}t) \\ &v = (1, \boldsymbol{v}) \Rightarrow v' = (1, \boldsymbol{v}') = (1, \boldsymbol{v} - \boldsymbol{U}) \\ &\partial_t = \partial_{t'} - \boldsymbol{U}\cdot\nabla' \qquad \nabla = \nabla' \end{aligned} \qquad (18)$$

hence $\partial_t + (\boldsymbol{v}\cdot\nabla) = \partial_{t'} + (\boldsymbol{v}'\cdot\nabla')$

where

$$\partial_t = \partial/\partial t$$

$$\nabla = (\partial_1, \partial_2, \partial_3)$$

$$\partial_k = \partial/\partial x^k$$

3.3 Infinitesimal Rotational Transformation

For later use, we consider a variable orthogonal transformation $\boldsymbol{R}$ and an associated infinitesimal transformation where an arbitrary vector $\boldsymbol{v}_0$ is sent to $\boldsymbol{v} = \boldsymbol{R}\boldsymbol{v}_0$. Suppose that a varied orthogonal transformation is written as $\boldsymbol{R}' = \boldsymbol{R} + \delta\boldsymbol{R}$ for an finitesimal variation $\delta\boldsymbol{R}$. We then have $\delta\boldsymbol{v} = \delta\boldsymbol{R}\boldsymbol{v}_0 = (\delta\boldsymbol{R})\boldsymbol{R}^{-1}\boldsymbol{v}$, so that we obtain the infinitely near vector $\boldsymbol{v} + \delta\boldsymbol{v}$ by the action of a general $\boldsymbol{R}$ as

$$\boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v} = \big(\boldsymbol{I} + (\delta\boldsymbol{R})\boldsymbol{R}^{-1}\big)\boldsymbol{v}$$

where $(\delta\boldsymbol{R})\boldsymbol{R}^{-1}$ is skew-symmetric for othogonal matrices $\boldsymbol{R}$ and $\boldsymbol{R} + \delta\boldsymbol{R}$**.

Analogously to Eq.(7), the infinitesimal gauge transformation is written as

$$\begin{aligned} \boldsymbol{R}(x) &= \exp[\boldsymbol{\theta}] = \boldsymbol{I} + \boldsymbol{\theta} + O(|\boldsymbol{\theta}|^2) = \boldsymbol{I} + \\ &\big(\boldsymbol{E}_1\theta^1 + \boldsymbol{E}_2\theta^2 + \boldsymbol{E}_3\theta^3\big) + O(|\boldsymbol{\theta}|^2) \end{aligned} \qquad (19)$$

where $\boldsymbol{R} \in SO(3)$, $|\boldsymbol{\theta}| \ll 1$, and $\boldsymbol{\theta} = \boldsymbol{E}_k\theta^k$ is a skew-symmetric 3×3 matrix. The second term $\boldsymbol{\theta}$ of Eq.(19) is an element of the algebra $so(3)$, and $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ are bases of $so(3)$

$$\boldsymbol{E}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad (20)$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Their commutation relations are given by

$$[\boldsymbol{E}_j, \boldsymbol{E}_k] = \epsilon_{jkl}\boldsymbol{E}_l \qquad (21)$$

which is equivalent to Eq.(8) if $\boldsymbol{E}_k$ is replaced by $\boldsymbol{\sigma}_k/2\mathrm{i}$.

Introducing an axial vector $\hat{\boldsymbol{\theta}}$ defined by $\hat{\boldsymbol{\theta}} = (\theta^1, \theta^2, \theta^3)^{\mathrm{T}}$, we have two equivalent expressions for multiplication with a vector $\boldsymbol{v} = (v^1, v^2, v^3)^{\mathrm{T}}$

$$\begin{aligned} &\boldsymbol{\theta}\boldsymbol{v} \ \text{(tensor multiplication)} = \\ &\hat{\boldsymbol{\theta}} \times \boldsymbol{v} \ \text{(vector product)} \end{aligned} \qquad (22)$$

In the analyses of this section 3, it is observed that key elements of framework of the gauge principle are there in fluid flows. Therefore, we might ask whether we have a gauge field already in Fluid Dynamics, in the subsequent sections.

4 FREE-FIELD LAGRANGIAN OF FLUID FLOWS

Suppose that a free-field Lagrangian of flows on a subdomain $M^3 \subset \mathcal{R}^3$ is given by***

$$\begin{aligned} \Lambda_f[\boldsymbol{u}, \rho, \phi] = &\int_{M^3} \mathrm{d}^3 x \Big[\rho(x)\Big(\frac{1}{2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle - \epsilon(\rho)\Big)\Big] + \\ &\int_{M^3} \mathrm{d}^3 x \ \phi(x)\Big[\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u})\Big] \end{aligned} \qquad (23)$$

where $\boldsymbol{u}(x,t) = (u^i)$ is the velocity field, ρ the fluid density, $\epsilon(\rho)$ the internal energy per unit mass (specific internal energy), $\phi(x)$ a scalar function. The first

* Second term of Eq.(16) is written in the form of total time derivative since $\langle \boldsymbol{v}, \boldsymbol{U}\rangle - \frac{1}{2}\boldsymbol{U}^2 = (\mathrm{d}/\mathrm{d}t)\Big(\langle \boldsymbol{x}(t), \boldsymbol{U}\rangle - \frac{1}{2}\boldsymbol{U}^2 t\Big)$. Therefore it is understood in Newtonian mechanics that this term does not play any role in the variational formulation.

** Since $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$ and $(\boldsymbol{R} + \delta\boldsymbol{R})(\boldsymbol{R} + \delta\boldsymbol{R})^{\mathrm{T}} = \boldsymbol{I}$ from (12), we have $(\delta\boldsymbol{R})\boldsymbol{R}^{-1} + (\boldsymbol{R}^{-1})^{\mathrm{T}}(\delta\boldsymbol{R})^{\mathrm{T}} = 0$.

*** This is the Lagrangian of Herivel[15] without CC Lin's constraint (Ref.[16, Sec.B], Ref.[23]).

integration of the Lagrangian is of the usual form generalized to a fluid flow, while the second integration term represents a constraint to satisfy the continuity condition with the velocity field $\boldsymbol{u}$ by using the scalar function $\phi(x)$ of Lagrange multiplier. It is assumed that the fluid is homentropic, i.e. entropy is constant, hence the internal energy ϵ is a function of density only. Then, the entropy is kept constant at all times due to the non-dissipative property of the ideal fluid.

It is seen from Section 3.2 that the first term of the Lagrangian Λ_f is regarded to be invariant with respect to Galilei transformation according to the prescription that $\Lambda_G \mathrm{d}t$ of Eq.(17) is replaced by $\Lambda_L^{(0)}\mathrm{d}t$ of Eq.(15). The second term is also verified to be invariant by using Eq.(18)

$$\partial_t \rho + (\boldsymbol{u}\cdot\mathrm{grad})\rho + \mathrm{div}\boldsymbol{u} \ \Rightarrow \partial_{t'}\rho' + (\boldsymbol{u}'\cdot\mathrm{grad}')\rho' + \mathrm{div}'\boldsymbol{u}' \quad \text{with} \quad \rho' = \rho(x')$$

The scalar function $\phi(x)$ (corresponding to a Lagrange multiplier) may be a different function $\phi'(x')$ in order to comply with the Galilei transformation.

Regarding the gauge transformation, it is already shown in Section 3.1 that the term $\int \mathrm{d}^3 x \rho \langle \boldsymbol{u}, \boldsymbol{u}\rangle$ is invariant with respect to both global and local gauge transformations by the rotation group $SO(3)$. In addition, the scalar term $\mathrm{div}(\rho\boldsymbol{u})$ can be shown to be invariant with respect to a global transformation by a fixed element of $SO(3)$, and other scalar functions are invarinat as well. Thus, the free-field Lagrangian Λ_f is invariant with respect to a global gauge transformation as well as Galilei transformation.

The variational principle is described with the action $\boldsymbol{\mathcal{A}}$ defined in terms of the Lagrangian Λ_f as follows

$$\delta\boldsymbol{\mathcal{A}} = 0$$

where

$$\boldsymbol{\mathcal{A}} = \int_{t_0}^{t_1} \Lambda_f[\boldsymbol{u}, \rho, \phi]\mathrm{d}t$$

Independent variations are taken for the field variables ϕ, $\boldsymbol{u}$, and ρ, that is[15,16]

$$\delta\Lambda_f = \int \mathrm{d}^3\xi \Big[\delta\phi(\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u})) + \delta\boldsymbol{u}\cdot(\rho\boldsymbol{u} - \rho\,\mathrm{grad}\phi) + \delta\rho\Big(\frac{1}{2}v^2 - h - \partial_t\phi - \boldsymbol{u}\cdot\mathrm{grad}\phi\Big)\Big] = 0$$

where $h = \epsilon + \rho\mathrm{d}\epsilon/\mathrm{d}\rho = \epsilon + p/\rho$ (since $\mathrm{d}\epsilon/\mathrm{d}\rho = p/\rho^2$ with a fixed entropy) is the specific enthalpy. Note that $\mathrm{d}h = (1/\rho)\mathrm{d}p$ in this case (p: the pressure). Thus, the variational principle, $\delta A = 0$ for independent arbitrary variations $\delta\phi$, $\delta\boldsymbol{v}$ and $\delta\rho$, leads to

$$\delta\phi: \quad \partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0 \tag{24}$$

$$\delta\boldsymbol{u}: \quad \boldsymbol{u} = \mathrm{grad}\phi \tag{25}$$

$$\delta\rho: \quad \frac{1}{2}\boldsymbol{v}^2 - h - \partial_t\phi - \boldsymbol{u}\cdot\mathrm{grad}\phi = -\Big(\partial_t\phi + \frac{1}{2}\boldsymbol{v}^2 + h\Big) = 0 \tag{26}$$

The first equation is just the continuity equation for a compressible fluid. The second describes that the velocity $\boldsymbol{u}$ has a potential ϕ, that is, the flow is a potential flow and the $\boldsymbol{u}$-field is irrotational. The third equation corresponds to an integral of the equation of motion. In fact, applying "grad" to Eq.(26), we obtain the Euler's equation of motion of a perfect fluid for a potential flow with $\boldsymbol{u} = \mathrm{grad}\phi$

$$\partial_t\boldsymbol{u} + \mathrm{grad}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\mathrm{grad}\, h \tag{27}$$

where

$$\mathrm{grad}\, h = \frac{1}{\rho}\,\mathrm{grad}\, p$$

Thus, as far as the action principle associated with Λ_f is concerned, we obtain the Euler's equation of motion for potential flows of a perfect fluid. It is interesting to recall that the flow of a superfluid (boson) in the ground state is described by a velocity potential. The fact that this equation describes the "ground state" is consistent with the Kelvin's theorem of minimum energy[24], which asserts that a potential flow has a minimum kinetic energy among all possible flows satisfying given conditions*.

Note that the left hand side (denoted by $\mathrm{D}_t\boldsymbol{u}$) of the Eq.(27) is the material time derivative for the potential flow velocity $u^k = \partial_k\phi$

$$\mathrm{D}_t\boldsymbol{u} := \partial_t\boldsymbol{u} + \mathrm{grad}\Big(\frac{1}{2}v^2\Big) = \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u} \tag{28}$$

In fact, we have $\partial_i(v^2/2) = v^k\partial_i v^k = (\partial_k\phi)\partial_i\partial_k\phi = (\partial_k\phi)\partial_k\partial_i\phi = v^k\partial_k v^i$.

Usually, in fluid mechanics, it is examined whether the equation derived from the variational

* Suppose that the velocity is represented as $\boldsymbol{v} = \boldsymbol{\nabla}\phi + \boldsymbol{v}'$ with div $\boldsymbol{v}' = 0$ and div $\boldsymbol{v} = \boldsymbol{\nabla}^2\phi$ (given), and that the normal component $\boldsymbol{n}\cdot\boldsymbol{v}'$ vanishes and $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{\nabla}\phi$ is given on the boundary surface S ($\boldsymbol{n}$ being a unit normal to S), where ϕ is a scalar function solving $\boldsymbol{\nabla}^2\phi = given$ with given values of $\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi$ on S. Then $\int(\boldsymbol{\nabla}\phi+\boldsymbol{v}')^2\mathrm{d}^3\boldsymbol{x} = \int(\boldsymbol{\nabla}\phi)^2\mathrm{d}^3\boldsymbol{x} + \int(\boldsymbol{v}')^2\mathrm{d}^3\boldsymbol{x} + 2\int_S \phi\boldsymbol{v}'\cdot\boldsymbol{n}\mathrm{d}S = \int(\boldsymbol{\nabla}\phi)^2\mathrm{d}^3\boldsymbol{x} + \int(\boldsymbol{v}')^2\mathrm{d}^3\boldsymbol{x} \geq \int(\boldsymbol{\nabla}\phi)^2\mathrm{d}^3\boldsymbol{x}$. This is a generalization to a compressible case.

method (i.e. Euler's equation of motion) is invariant with respect to Galilei transformation. In fact, the Eq.(27) is invariant with the Galilei transformation (18): $\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{U}$, $\partial_t = \partial_{t'} - \boldsymbol{U} \cdot \mathrm{grad}'$, since $\mathrm{grad} = \mathrm{grad}'$, and

$$\begin{aligned} \mathrm{D}_t \boldsymbol{u} &= \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \mathrm{grad})\boldsymbol{u} \\ &= (\partial_{t'} - \boldsymbol{U} \cdot \mathrm{grad}')\boldsymbol{u}' + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \mathrm{grad}'\boldsymbol{u}' \\ &= \partial_{t'}\boldsymbol{u}' + (\boldsymbol{u}' \cdot \mathrm{grad}')\boldsymbol{u}' \\ &= \partial_{t'}\boldsymbol{u}' + \mathrm{grad}'\left(\frac{1}{2}(\boldsymbol{v}')^2\right) = \mathrm{D}'_t \boldsymbol{u}' \end{aligned} \tag{29}$$

The relation $\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{U}$ and $\boldsymbol{u} = \mathrm{grad}\phi$ require $\phi(x') = \phi'(\boldsymbol{x}') + \boldsymbol{U} \cdot \boldsymbol{x}'$.

It is noted that local gauge transformation of the velocity field $\boldsymbol{u}'(x) = \boldsymbol{R}(x)\boldsymbol{u}(x)$ gives rise to a rotational component in the velocity field, as is shown in the next section. The variation considered in this section can represent only irrotational velocity field. In order to resolve this incompleteness, a gauge-covariant derivative is defined in the next section by introduing a gauge field, and in Section 6 a complete variational formulation is proposed on the basis of the Hamilton's principle in terms of material variations and the gauge principle.

5 LOCAL GAUGE TRANSFORMATION

5.1 Infinitesimal Rotational Transformation

Let us consider an important consequence of the local infinitesimal gauge transformation (19): $\boldsymbol{R}(x) = \exp[\boldsymbol{\theta}(x)] \approx \boldsymbol{I} + \boldsymbol{\theta} = \boldsymbol{I} + (\boldsymbol{E}_1\theta^1 + \boldsymbol{E}_2\theta^2 + \boldsymbol{E}_3\theta^3)$, where $\boldsymbol{\theta} \in SO(3)$, and $|\boldsymbol{\theta}| \ll 1$. According to Eq.(20), the transformation matrix $\boldsymbol{\theta} = \boldsymbol{E}_k\theta^k$ is a skew-symmetric 3×3 matrix. Representing the multiplication $\boldsymbol{\theta u}$ in the form of a vector product according to Eq.(22), the velocity $\boldsymbol{u}$ is transformed as

$$\begin{aligned} \boldsymbol{u}(x) \to \boldsymbol{u}' &= \boldsymbol{R}(x)\boldsymbol{u}(x) \approx \boldsymbol{u} + \boldsymbol{\theta u} \\ &= \boldsymbol{u} + \hat{\boldsymbol{\theta}} \times \boldsymbol{u} \end{aligned} \tag{30}$$

It is remarkable that the transformed field $\boldsymbol{u}' = \boldsymbol{R}(x)\boldsymbol{u}(x)$ is rotational even if $\boldsymbol{u}$ is irrotational. In fact, one can represent the second term in terms of a vector potential $\boldsymbol{B}$ and a scalar potential f defined by

$$\hat{\boldsymbol{\theta}} \times \boldsymbol{u} = \mathrm{curl}\, \boldsymbol{B} + \mathrm{grad}\, f$$

together with the gauge condition, $\mathrm{div}\, \boldsymbol{B} = 0$. Taking curl of $\hat{\boldsymbol{\theta}} \times \boldsymbol{u}$, we have

$$\mathrm{curl}\,(\hat{\boldsymbol{\theta}} \times \boldsymbol{u}) = \mathrm{curl}\,(\mathrm{curl}\, \boldsymbol{B}) = -\boldsymbol{\Delta B}$$

where $\boldsymbol{\Delta}$ is the Laplacian. The vector potential $\boldsymbol{B}$ is determined by the equation

$$\begin{aligned} \boldsymbol{\Delta B} = (\hat{\boldsymbol{\theta}} \cdot \mathrm{grad})\, \boldsymbol{u} + (\mathrm{div}\, \hat{\boldsymbol{\theta}})\boldsymbol{u} - \\ (\boldsymbol{u} \cdot \mathrm{grad})\, \hat{\boldsymbol{\theta}} - (\mathrm{div}\, \boldsymbol{u})\hat{\boldsymbol{\theta}} \end{aligned}$$

Thus, it is found that the gauge transformation introduces a rotational component in the velocity field even if $\boldsymbol{u}$ is irrotational.*

Henceforth, the velocity vector is denoted by $\boldsymbol{v}$ instead of $\boldsymbol{u}$ in order to emphasize explicitly that the vector $\boldsymbol{v}$ denotes a velocity field of a material particle, which is inevitably rotational.

5.2 Gauge Principle in Dynamical Systems

The Lagrangian Λ_f of Eq.(23) has been shown to be invariant under a global $SO(3)$ gauge transformation. We now require that not only the Lagrangian (defined in the next section) but its variational form should be invariant under local gauge transformations.

In addition, it was noted in the introduction that the gauge field $\boldsymbol{\Omega}$ of dynamical systems, such as in the $d = 1$ field theory of the model of a nuclear rotation[13], is defined only for the covariant derivative with respect to the time t. This means that the replacement in the present fluid flows would be of the form, $\mathrm{D}_t \to \nabla_t = \mathrm{D}_t + \boldsymbol{\Omega}(x)$ where D_t defined by Eq.(28) denotes the material derivative of a potential flow, while the spatial derivatives are unchanged $\partial_k \to \partial_k$ $(k = 1, 2, 3)$.

According to the scenario of gauge theory (e.g. [12,13]), the velocity field $\boldsymbol{v}$ and the covariant derivative $\mathrm{D}_t\boldsymbol{v}$ should obey the following the transformation laws

$$\boldsymbol{v} \mapsto \boldsymbol{v}' = \exp[\boldsymbol{\theta}(t, x)]\, \boldsymbol{v} \tag{31}$$

$$\nabla_t \boldsymbol{v} \mapsto \nabla'_t \boldsymbol{v}' = \exp[\boldsymbol{\theta}(t, x)]\, \nabla_t \boldsymbol{v} \tag{32}$$

where the covariant derivative is defined as follows

$$\nabla_t \boldsymbol{v} := \mathrm{D}_t \boldsymbol{v} + \boldsymbol{\Omega v}$$

* The new component signifies rotational motion of fluid, i.e. local rotation of fluid material. If particles are equivalent and indistinguishable such as the superfluid He^4, the rotational motion would not be captured. The superfluid He^4 obeys Bose-Einstein statistics in which particles are equivalent and indistinguishable. Therefore the flow would be irrotational. This is not the case when we consider the motion of a fluid composed of distinguishable particles such as in an ordinary fluid. Local rotation is distinguishable and there must be a formal structure to take into account the local rotation of fluid particles.

$$= \partial_t \boldsymbol{v} + \mathrm{grad}(\boldsymbol{v}^2/2) + \hat{\boldsymbol{\Omega}} \times \boldsymbol{v} \tag{33}$$

and $\boldsymbol{\theta}, \boldsymbol{\Omega} \in SO(3)$, i.e. $\boldsymbol{\theta}$ and $\boldsymbol{\Omega}$ are skew-symmetric matrices*, where $\hat{\boldsymbol{\Omega}}$ is the axial vector counterpart of $\boldsymbol{\Omega}$. From the above Eqs.(31)∼(33), it is found that the gauge field operator $\boldsymbol{\Omega}$ is transformed as

$$\boldsymbol{\Omega} \to \boldsymbol{\Omega}' = \mathrm{e}^{\boldsymbol{\theta}} \boldsymbol{\Omega} \mathrm{e}^{-\boldsymbol{\theta}} - \left(\partial_t \mathrm{e}^{\boldsymbol{\theta}}\right)\mathrm{e}^{-\boldsymbol{\theta}} \tag{34}$$

Corresponding to the infinitesimal transformation, we have the expansion, $\mathrm{e}^{\boldsymbol{\theta}} = 1 + \boldsymbol{\theta} + (|\boldsymbol{\theta}|^2)$. Using $\delta\boldsymbol{\theta}$ instead of $\boldsymbol{\theta}$

$$\boldsymbol{v} \to \boldsymbol{v}' = \left(1 + \delta\boldsymbol{\theta}\right)\boldsymbol{v} = \boldsymbol{v} + \delta\hat{\boldsymbol{\theta}} \times \boldsymbol{v} \tag{35}$$

up to the first order, and the gauge field $\hat{\boldsymbol{\Omega}}$ (in vector form) is transformed as

$$\hat{\boldsymbol{\Omega}} \to \hat{\boldsymbol{\Omega}}' = \hat{\boldsymbol{\Omega}} + \delta\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\Omega}} - \partial_t(\delta\hat{\boldsymbol{\theta}}) \tag{36}$$

The second term on the right hand side came from $\delta\boldsymbol{\theta}\boldsymbol{\Omega} - \boldsymbol{\Omega}\delta\boldsymbol{\theta}$. This is equivalent to the non-Aberian transformation law (10) for the Yang-Mills gauge field $\boldsymbol{A}_\mu$ if $2q\boldsymbol{A}_\mu$ is replaced by $\boldsymbol{\Omega}$ and $2q\boldsymbol{\alpha}$ by $-\delta\boldsymbol{\theta}$ only for $\mu = t$.

5.3 Galilei Invariance

We require that the covariant derivative (33) is invariant with respect to a Galilei transformation. Applying Eq.(18) with using ($_*$) insteaf of ($'$) to denote Galilei-transformed variables (e.g. $\boldsymbol{v}' = \boldsymbol{v}_*$ is written as $\boldsymbol{v} - \boldsymbol{U}$), the covariant derivative $\nabla_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\boldsymbol{v}^2/2) + \hat{\boldsymbol{\Omega}} \times \boldsymbol{v}$ is transformed to

$$\begin{aligned}
&(\partial_{t_*} - \boldsymbol{U}\cdot\nabla_*)(\boldsymbol{v}_* + \boldsymbol{U}) + \nabla_* \frac{1}{2}|\boldsymbol{v}_* + \boldsymbol{U}|^2 + \\
&\qquad \hat{\boldsymbol{\Omega}} \times (\boldsymbol{v}_* + \boldsymbol{U}) = \partial_{t_*}\boldsymbol{v}_* + \nabla_*(\boldsymbol{v}_*^2/2) + \hat{\boldsymbol{\Omega}} \times \boldsymbol{v}_* - \\
&\qquad (\boldsymbol{U}\cdot\nabla_*)\boldsymbol{v}_* + \hat{\boldsymbol{\Omega}} \times \boldsymbol{U} + \nabla_*\left(\boldsymbol{v}_* \cdot \boldsymbol{U}\right)
\end{aligned}$$

since $\nabla_*(U^2) = 0$. We expect that the right hand side is equal to $\partial_{t_*}\boldsymbol{v}_* + \nabla_*(\boldsymbol{v}_*^2/2) + \hat{\boldsymbol{\Omega}}_* \times \boldsymbol{v}_*$, which requires

$$\begin{aligned}
0 = {} & (\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}_*) \times \boldsymbol{v}_* - (\boldsymbol{U}\cdot\nabla_*)\boldsymbol{v}_* + \\
&\qquad \hat{\boldsymbol{\Omega}} \times \boldsymbol{U} + \nabla_*\left(\boldsymbol{v}_* \cdot \boldsymbol{U}\right) = (\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}_*) \times \boldsymbol{v}_* + \\
&\qquad (\hat{\boldsymbol{\Omega}} - \nabla_* \times \boldsymbol{v}_*) \times \boldsymbol{U}
\end{aligned}$$

where the following vector identity is used: $\boldsymbol{U} \times (\nabla_* \times \boldsymbol{v}_*) = -(\boldsymbol{U}\cdot\nabla_*)\boldsymbol{v}_* + \nabla_*(\boldsymbol{U}\cdot\boldsymbol{v}_*)$ with a constant $\boldsymbol{U}$. The last expression vanishes identically, if

$$\begin{aligned}
&\hat{\boldsymbol{\Omega}} = \nabla \times \boldsymbol{v} \\
&\hat{\boldsymbol{\Omega}}_* = \nabla_* \times \boldsymbol{v}_* = \nabla \times \boldsymbol{v} = \hat{\boldsymbol{\Omega}}
\end{aligned} \tag{37}$$

Thus, the Galilei invariance of $\nabla_t \boldsymbol{v}$ requires that the gauge field $\hat{\boldsymbol{\Omega}}$ coincides with the vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

and the covariant derivative $\nabla_t \boldsymbol{v}$ is represented by

$$\begin{aligned}
\nabla_t \boldsymbol{v} &= \partial_t \boldsymbol{v} + \nabla\left(\frac{1}{2}\boldsymbol{v}^2\right) + \boldsymbol{\omega} \times \boldsymbol{v} \\
&= \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}
\end{aligned} \tag{38}$$

which is usually called as the material time derivative or Lagrange derivative of $\boldsymbol{v}$. It is interesing to observe that this expression of gauge-covariant derivative $\nabla_t \boldsymbol{v}$ is consistent with that of geometrically-covariant derivative (see Ref.[25, §7]).

6 HAMILTON'S PRINCIPLE UNDER ISENTROPIC MATERIAL VARIATIONS

We return to the variational principle again. From the analysis based on the local gauge invariance, we have arrived at the covariant derivative (38), which is the material time derivative of velocity. This suggests that the variation should take into account displacement or motion of individual particles[17,18]. In other words, the gauge invariance requires that laws of fluid motion should be expressed in a form equivalent to every individual particle. In order to comply with the local gauge invariance, we carry out material variations according to the following scenario consisting of three conditions:

(1) Kinematic condition: All the variations are taken so as to follow the trajectories of material particles, and all the mathematical expressions are given for a fixed mass element defined by Eq.(39) below. As a consequence, the equation of continuity must be satisfied always. This is given by Eq.(41).

(2) Condition of physical material: An ideal fluid is defined by the property that there is no dissipative mechanism within it such as dissipations caused by viscosity or thermal conductivity (see Ref.[22, §2]). As a consequence (see Ref.[22, §49]), the entropy s per unit mass (i.e. specific entropy) remains constant along the motion of each material particle, namely isentropic, which is expressed by Eq.(43). Here, the fluid is not necessarily assumed to be homentropic.

(3) Gauge condition: All the expressions of the formualtion must satisfy both global and local gauge invariance. Therefore, not only the action functional

* The property $\boldsymbol{\theta}, \boldsymbol{\Omega} \in SO(3)$ means that we are considering the principal fiber bundle.

$\boldsymbol{\mathcal{A}} = \int \boldsymbol{\Lambda}_{\text{fluid}}\text{d}t$, but its varied form must be gauge-invariant, and the gauge-covariant derivative $\boldsymbol{\nabla}_t$ of Eq.(38) should be used for the variation, where the Lagrangian $\boldsymbol{\Lambda}_{\text{fluid}}$ is defined below by Eq.(48) newly for flows of an ideal fluid. In this Lagrangian, there is no additional constraint term such as the second term of Eq.(23).

In the following, we consider an isentropic material variation according to the above scenario. As a result of the variational formulation satisfying local gauge invariance, we will obtain the Euler's equation of motion for an ideal fluid. Fortunately, there are some byproducts of the present formulation. From the global gauge invariance of $SO(3)$, we will obtain a Noether's conservation law, which is found to be the conservation of total angular momentum of the system. In addition, the Lagrangian has a symmetry with respect to particle permutation, which leads to a local law of vorticity conservation, i.e. the vorticity equation. Thus, it is found that the well-known equations are related to symmetries of the fluid system.

6.1 Flows of an Ideal Fluid

Suppose that individual fluid particles are distinguished by the coordinates $\boldsymbol{a} = (a^i) = (a, b, c) \in \mathcal{R}^3$, and that their motion is described by a flow $\boldsymbol{\xi}_\tau$ (map) which takes a particle located at $\boldsymbol{\xi}_0\boldsymbol{a}$ when $\tau = 0$ to the position $\boldsymbol{\xi}(\tau, \boldsymbol{a}) = \boldsymbol{\xi}_\tau\boldsymbol{a}$ at a time τ. Suppose that the coordinate parameters (a, b, c) remain constant following the motion of the fluid paticles, then the particle velocity is given by $\boldsymbol{v} = \partial_\tau\boldsymbol{\xi}(\tau, \boldsymbol{a}) = \boldsymbol{v}(t, \boldsymbol{x})$, where $\partial_\tau = \partial/\partial\tau^*$. Consider a mass of fluid contained in a volume M_τ moving with the material particles. All the particles $\forall\boldsymbol{a} \in A$ (where $A \subset \mathcal{R}^3$) are mapped to $\boldsymbol{\xi}(\tau, \boldsymbol{a})$, which composes a moving mass M_τ. It is convenient to assign these coordinates (a, b, c) so that the three-form $\text{d}^3\boldsymbol{a} = \text{d}a\text{d}b\text{d}c$ represents the mass in a volume element $\text{d}^3\boldsymbol{\xi} = \text{d}x\text{d}y\text{d}z$

$$\text{d}^3m = \text{d}^3\boldsymbol{a} = \text{d}a\text{d}b\text{d}c \tag{39}$$

Hence, $\boldsymbol{a} = (a, b, c)$ is considered a mass-coordinate. Then the mass density ρ at $\boldsymbol{\xi}(t, \boldsymbol{a}) = (x, y, z)$ is expressed by $\text{d}^3m = \rho\text{d}^3\boldsymbol{\xi}$ and also given by the inverse of the Jacobian $J = \partial(\boldsymbol{x})/\partial(\boldsymbol{a})$ of the map

$$\text{d}^3m = \rho\text{d}^3\boldsymbol{\xi}$$

$$\rho = \frac{\partial(a, b, c)}{\partial(x, y, z)} = J^{-1} \tag{40}$$

$$J = \frac{\partial(x, y, z)}{\partial(a, b, c)}$$

Rate of change of a form F along a particle motion is represented by the Lie derivative $\mathcal{L}(F)$. The mass elelment is a three-form $F^3 = \text{d}^3m = \rho\text{d}^3\boldsymbol{\xi}$. Then, invariance of a mass along the particle trajectory is represented by

$$0 = \frac{\partial}{\partial\tau}(\text{d}^3m) = \mathcal{L}_{\text{X}}(\text{d}^3\boldsymbol{m}) = \mathcal{L}_{\text{X}}(\rho\text{d}^3\boldsymbol{\xi})$$

$$= [\partial_t\rho + \text{div}(\rho\boldsymbol{v})]\text{d}^3\boldsymbol{\xi} \tag{41}$$

where $\mathcal{L}_{\text{X}}$ is the Lie derivative with the velocity vector $\text{X} = \partial_t + v^i(t, \boldsymbol{x})\partial_i$**. Therefore, the second integration term of Eq.(23) identically vanishes. The Eq.(41) is equivalent to the continuity equation, and rewritten as

$$\frac{\text{d}\rho}{\text{d}t} := (\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{42}$$

where $\text{d}/\text{d}t = \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$. The isentropic motion is represented by $\mathcal{L}_{\text{X}}(s\rho\text{d}^3\boldsymbol{\xi}) = 0$. Recalling the following identity

$$\partial_t(s\rho) + \text{div}(s\rho\boldsymbol{v}) = \rho(\partial_t s + (\boldsymbol{v}\cdot\boldsymbol{\nabla})s) + s(\partial_t\rho + \text{div}(\rho\boldsymbol{v}))$$

and using Eq.(41), we obtain the equation of entropy conservation

$$\text{d}s/\text{d}t = \partial_t s + \boldsymbol{v}\cdot\boldsymbol{\nabla}s = 0 \tag{43}$$

The invariance of $\boldsymbol{a}$ following the fluid motion of the velocity field $\boldsymbol{v}$ is expressed by

$$\begin{aligned} \text{d}a^i/\text{d}t &= \partial_t a^i + (\boldsymbol{v}\cdot\boldsymbol{\nabla})a^i = 0 \\ \boldsymbol{a} &= (a^1, a^2, a^3) = (a, b, c) \end{aligned} \tag{44}$$

Henceforth, we consider an isentropic material variation satisfying Eqs.(42)~(44).

6.2 Lagrangian and Variational Formulation

Let a set of varied particle-trajectories be given by $\hat{\boldsymbol{\xi}}(\tau, \boldsymbol{a} : \varepsilon)$ for $\varepsilon \in (-1, 1)$, where each value of ε denotes a single varied trajectory of all material particles of $\forall\boldsymbol{a} \in A$ and the initial mass M_0 (or A) is chosen arbitrary within fluid. The trajectory $\varepsilon = 0$ is the one $\boldsymbol{\xi}_\tau\boldsymbol{a} = \boldsymbol{\xi}(\tau, \boldsymbol{a})$ to be investigated. We consider a particular set of trajectories $\hat{\boldsymbol{\xi}}(\tau, \boldsymbol{a} : \varepsilon)$ given by

$$\hat{\boldsymbol{\xi}}(\tau, \boldsymbol{a} : \varepsilon) = \boldsymbol{\xi}_\tau\boldsymbol{a} + \varepsilon\boldsymbol{\eta}(\boldsymbol{\xi}_\tau\boldsymbol{a}) \qquad \text{for} \quad \forall\boldsymbol{a} \in A$$

* The time coordinate used in combination with the Lagrangian particle coordinates $\boldsymbol{a}$ is denoted by τ. Inverse map of $\boldsymbol{x} = \xi(\tau, \boldsymbol{a})$ is $\boldsymbol{a} = \boldsymbol{a}(t, \boldsymbol{x})$, where $t = \tau$, $\boldsymbol{x} = (x, y, z)$ are Eulerian coordinates.

** Lie derivative $\mathcal{L}_{\text{X}}$ is the derivative as one moves along the trajectory generated by a tangent vector $\text{X} = \partial_t + v^i(t, \boldsymbol{x})\partial_i$, i.e. the derivative $\partial/\partial\tau$ with $\boldsymbol{a}$ fixed. Hence, local mass conservation is given by $\mathcal{L}_{\text{X}}(\rho\text{d}^3\boldsymbol{\xi}) = [\partial_t\rho + \text{div}(\rho\boldsymbol{v})]\text{d}^3\boldsymbol{\xi} = 0$ for derivatives of forms.

The variation vector field $\boldsymbol{\eta}(\tau)$ is constrained to vanish at some endpoints (e.g. at t_0 and t_1) of the trajectory as well as on the moving boundary surface ∂M_τ of M_τ, namely

$$\boldsymbol{\eta}(t_0) = 0 \qquad \boldsymbol{\eta}(t_1) = 0 \quad \text{for any} \ \ \boldsymbol{\xi}_\tau \in M_\tau \tag{45}$$

$$\boldsymbol{\eta}(\boldsymbol{\xi}_S) = 0 \quad \text{for any} \ \ t \ \text{ and } \ \text{at any} \ \ \boldsymbol{\xi}_S \in \partial M_\tau \tag{46}$$

where each boundary point $\boldsymbol{\xi}_S \in \partial M_\tau$ moves with the fluid particle. Differentiating $\hat{\boldsymbol{\xi}}(\tau, \boldsymbol{a} : \varepsilon)$ with respect to ε with $\boldsymbol{a}$ fixed

$$\frac{\partial}{\partial \varepsilon}\hat{\boldsymbol{\xi}}(\tau, \boldsymbol{a} : \varepsilon)\big|_{\varepsilon=0,\boldsymbol{a},\tau:\,\text{fixed}} = \boldsymbol{\eta}(\boldsymbol{\xi}_\tau \boldsymbol{a}) \qquad (:= \delta\boldsymbol{\xi}) \tag{47}$$

which is classically called a virtual displacement and written as $\delta\boldsymbol{\xi}$.

Suppose that the Lagrangian of flows of an ideal fluid is defined by

$$\boldsymbol{\Lambda}_{\text{fluid}}[\boldsymbol{\xi}_\tau \boldsymbol{a}] = \int_{M_\tau} \frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathrm{d}^3\boldsymbol{\xi} - \int_{M_\tau} e(\rho)\rho(\boldsymbol{\xi})\mathrm{d}^3\boldsymbol{\xi} \qquad (\forall \boldsymbol{a} \in A) \tag{48}$$

$e = e(\rho, s)$ is the specific internal energy with s being the specific entropy. The Lagrangian $\boldsymbol{\Lambda}_{\text{fluid}}$ is of a form generalized to a macroscopic continuous medium of a fluid. The first and second integral are written as I_1 and I_2 respectively.

The variational principle, namely the Hamilton's principle, is described by $\delta\boldsymbol{\mathcal{A}} = 0$, where the action functional is defined by

$$\begin{aligned}\boldsymbol{\mathcal{A}} &= \int_{t_0}^{t_1} \boldsymbol{\Lambda}_{\text{fluid}}[\boldsymbol{\xi}_\tau \boldsymbol{a} + \varepsilon\boldsymbol{\eta}]\mathrm{d}\tau \\ &= \int_{t_0}^{t_1} \big(I_1[\boldsymbol{\xi}_\tau \boldsymbol{a} + \varepsilon\boldsymbol{\eta}] - I_2[\boldsymbol{\xi}_\tau \boldsymbol{a} + \varepsilon\boldsymbol{\eta}]\big)\mathrm{d}\tau\end{aligned}$$

Taking variation with respect to ε with $\boldsymbol{a}$ fixed, i.e. $\delta = (\partial/\partial\varepsilon)|_{\varepsilon=0}$, the variational principle reads

$$\delta\boldsymbol{\mathcal{A}} = \int_{t_0}^{t_1} (\delta I_1 - \delta I_2)\mathrm{d}\tau = 0 \tag{49}$$

6.3 Material Variation

First, we consider variation of the second integral I_2

$$\begin{aligned}\delta I_2 &= \delta \int_{M_\tau} e(\rho, s)\rho(\boldsymbol{\xi})\mathrm{d}^3\boldsymbol{\xi} \\ &= \int_{M_\tau} \delta e \ \rho\mathrm{d}^3\boldsymbol{\xi} + \int_{M_\tau} e\delta\big(\rho\mathrm{d}^3\boldsymbol{\xi}\big)\end{aligned} \tag{50}$$

The variation $\delta\big(\rho\mathrm{d}^3\boldsymbol{\xi}\big)$ is given by

$$\delta\big(\rho\mathrm{d}^3\boldsymbol{\xi}\big) = (\delta\rho + \rho \ \mathrm{div}\boldsymbol{\eta})\mathrm{d}^3\boldsymbol{\xi} \tag{51}$$

This is obtained from Eqs.(41) and (42), because the Eq.(47) is observed as describing that $\boldsymbol{\eta}$ is the velocity at a time $\varepsilon = 0$ of a motion for which the parameter ε plays the role of time, where the convective derivative $\mathrm{d}/\mathrm{d}t$ and velocity $\boldsymbol{v}$ in Eq.(42) are replaced by δ and $\boldsymbol{\eta}$, respectively. Here, we take the material variation with keeping $\boldsymbol{a}$ fixed, i.e. $\delta(\rho\mathrm{d}^3\boldsymbol{\xi}) = 0$. Hence, the variation is constraint to satisfy

$$(\delta\rho + \rho\mathrm{div}\boldsymbol{\eta})\mathrm{d}^3\boldsymbol{\xi} = \mathcal{L}_Y(\rho\mathrm{d}^3\boldsymbol{\xi}) = \mathcal{L}_Y(\mathrm{d}^3 m) = 0 \qquad (Y = \partial_\varepsilon + \eta^i\partial_i) \tag{52}$$

Thus, the second term of Eq.(50) vanishes.

Using the δ-operation, the isentropic variation is given by $\delta s = 0$. Then, using a thermodynamic relation $\partial e/\partial\rho = p/\rho^2$ and Eq.(52), we have

$$\delta e = \frac{\partial e}{\partial \rho}\delta\rho + \frac{\partial e}{\partial s}\delta s = \frac{p}{\rho^2}\delta\rho = -\frac{p}{\rho}\mathrm{div}\boldsymbol{\eta} \tag{53}$$

Thus we obtain

$$\begin{aligned}\delta I_2 &= -\int_{M_\tau} p\mathrm{div}\boldsymbol{\eta}\mathrm{d}^3\boldsymbol{\xi} \\ &= -\int_{M_\tau} \mathrm{div}\big(p\boldsymbol{\eta}\big)\mathrm{d}^3\boldsymbol{\xi} + \int_{M_\tau} (\boldsymbol{\eta}\cdot\mathrm{grad})p\mathrm{d}^3\boldsymbol{\xi} \\ &= -\int_{\partial M_\tau} p\langle\boldsymbol{\eta}(\boldsymbol{\xi}_S), \boldsymbol{n}\rangle\mathrm{d}^2 S + \int_{M_\tau} \langle\mathrm{grad}p, \boldsymbol{\eta}\rangle\mathrm{d}^3\boldsymbol{\xi}\end{aligned} \tag{54}$$

where $\boldsymbol{n}$ is a unit outward normal to the bounding surface ∂M_τ.

Next, we consider variation of the first integral I_1

$$\begin{aligned}\delta I_1 &= \delta \int_{M_\tau} \frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle\rho(x)\mathrm{d}^3\boldsymbol{\xi} \\ &= \int_{M_\tau} \langle \boldsymbol{v}, \delta\boldsymbol{v}\rangle\rho\mathrm{d}^3\boldsymbol{\xi} + \int_{M_\tau} \frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle\delta\big(\rho\mathrm{d}^3\boldsymbol{\xi}\big)\end{aligned} \tag{55}$$

The second term vanishes as before. Regarding the first term, we use the expression $\rho\mathrm{d}^3\boldsymbol{\xi} = \mathrm{d}m(\boldsymbol{a})$, emphasizing that $\mathrm{d}m$ is an invariant mass element. In addition, we take the velocity variation with keeping $\boldsymbol{a}$ fixed

$$\delta\boldsymbol{v} = \delta\big(\partial_\tau\hat{\boldsymbol{\xi}}\big) = \partial_\tau\big(\delta\hat{\boldsymbol{\xi}}\big) = \partial_\tau\boldsymbol{\eta}$$

Furthermore, the following equation is used

$$\langle \boldsymbol{v}, \delta \boldsymbol{v}\rangle \mathrm{d}m = \langle \boldsymbol{v}, \partial_\tau \boldsymbol{\eta}\rangle \mathrm{d}m = \frac{\partial}{\partial \tau}\langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \mathrm{d}m - \langle \boldsymbol{\nabla}_t \boldsymbol{v}, \boldsymbol{\eta}\rangle \mathrm{d}m \tag{56}$$

where $\partial_\tau(\mathrm{d}m) = 0$, and the covariant derivative $\boldsymbol{\nabla}_t \boldsymbol{v}$ defined by Eq.(33) is substituted in the last term in accordance with the gauge principle. This is consistent with the fact that the derivative $\partial_\tau \boldsymbol{v}$ (with $\boldsymbol{a}$ fixed) is equivalent to rate of change with respect to a fixed material particle and should be given by the covariant derivative Eq.(33). It can be shown without difficulty that the Eq.(56) satisfies the local gauge invariance. Then, we have*

$$\delta I_1 = \int_{M_\tau} \langle \boldsymbol{v}, \delta \boldsymbol{v}\rangle \mathrm{d}m = \frac{\partial}{\partial \tau}\int_{M_\tau} \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} - \int_{M_\tau} \langle \boldsymbol{\nabla}_t \boldsymbol{v}, \boldsymbol{\eta}\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} \tag{57}$$

Substituting Eqs.(54) and (57) in Eq.(49), we obtain

$$\int_{M_{t_1}} \langle \boldsymbol{v}(t_1), \boldsymbol{\eta}(t_1)\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} - \int_{M_{t_0}} \langle \boldsymbol{v}(t_0), \boldsymbol{\eta}(t_0)\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} + \int_{t_0}^{t_1} \mathrm{d}\tau \int_{\partial M_\tau} p\langle \boldsymbol{\eta}(\boldsymbol{x}_S), \boldsymbol{n}\rangle \mathrm{d}S - \int_{t_0}^{t_1} \mathrm{d}\tau \cdot \int_{M_\tau} \left\langle \left(\boldsymbol{\nabla}_t \boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla} p\right), \boldsymbol{\eta}\right\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} = 0 \tag{58}$$

First, three terms vanish due to the constraints of the variation field $\boldsymbol{\eta}$ of Eqs.(45) and (46). The only remaining expression, the last term, must vanish for arbitrary variation vector $\boldsymbol{\eta}$. Thus, the following equation is derived from the variational principle

$$\boldsymbol{\nabla}_t \boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla} p = 0 \tag{59}$$

This is the Euler's equation of motion itself. In fact, using Eq.(38) for $\boldsymbol{\nabla}_t \boldsymbol{v}$, we have

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\boldsymbol{\nabla} h \tag{60}$$

or equivalently, using the last expression of Eq.(38) and $\boldsymbol{\nabla} h = (1/\rho)\boldsymbol{\nabla} p$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p \tag{61}$$

This is to be supplemented with the equations of continuity (43) and entropy (43).

The present variational formulation using the covariant derivative $\boldsymbol{\nabla}_t \boldsymbol{v}$ is natural in the following sense that, under the boundary constraints of Eqs.(45) and (46) together with the mass conservation (52) and entropy conservation $\delta s = 0$, no additional condition is imposed to $\boldsymbol{\Lambda}_{\mathrm{fluid}}$ to derive the equations of motion of an ideal fluid.

The form of Lagrangian (48) is compact with no constraint term, and the variation is carried out adiabatically by following particle trajectories. In the conventional variations[16,19], the Lagarangian has additional constraint terms which are imposed to obtain rotational component of velocity field**, or the covariant derivative such as Eq.(38) is used as a prerequisite identity for the variation[18].

6.4 Noether's Theorem

Global gauge transformation of a vector $\boldsymbol{v}$ is represented by $\boldsymbol{v}(\boldsymbol{\xi}) \mapsto \boldsymbol{v}'(\boldsymbol{\xi}) = \boldsymbol{R}\boldsymbol{v}(\boldsymbol{\xi})$ with a fixed element $\boldsymbol{R} \in SO(3)$ at every point $\boldsymbol{\xi} \in M_\tau$, which is nothing else than a uniform rotation. Consider an infinitesimal uniform rotation vector $\delta\hat{\boldsymbol{\theta}} = |\delta\hat{\boldsymbol{\theta}}|\boldsymbol{e}$, which is an infinitesimal rotation of the angle $|\delta\hat{\boldsymbol{\theta}}|$ about the axis in the direction denoted by a unit vector $\boldsymbol{e}$. Then, an infinitesimal global gauge transformation is defined for the position vector as

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + \delta\hat{\boldsymbol{\theta}} \times \boldsymbol{\xi} \tag{62}$$

Next, consider an infinitesimal variation of the Lagrangian $\boldsymbol{\Lambda}_{\mathrm{fluid}}$ due to this transformation. From Eqs.(57) and (54), the variation is given by

$$\delta\boldsymbol{\Lambda}_{\mathrm{fluid}} = \frac{\partial}{\partial \tau}\int_{M_\tau} \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} + \int_{\partial M_\tau} p\langle \boldsymbol{\eta}, \boldsymbol{n}\rangle \mathrm{d}^2 S - \int_{M_\tau} \left\langle \left(\boldsymbol{\nabla}_t \boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla} p\right), \boldsymbol{\eta}\right\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} \tag{63}$$

where $\boldsymbol{\eta} = \delta\hat{\boldsymbol{\theta}} \times \boldsymbol{\xi}$ with a constant vector $\delta\hat{\boldsymbol{\theta}}$, and $\boldsymbol{n}$ is a unit outward normal to ∂M_τ. The last term vanishes owing to the equation of motion (59). The Noether's theorem for the conserved current is given by $\delta\boldsymbol{\Lambda}_{\mathrm{fluid}} = 0$, which is expressed by

$$\frac{\partial}{\partial \tau}\int_{M_\tau} \langle \boldsymbol{v}, \boldsymbol{\eta}(\boldsymbol{\xi})\rangle \rho \mathrm{d}^3 \boldsymbol{\xi} + \int_{\partial M_\tau} p\langle \boldsymbol{\eta}(\boldsymbol{\xi}_S), \boldsymbol{n}\rangle \mathrm{d}^2 S = 0 \tag{64}$$

* Regarding the first term on the right of Eq.(57), $\int_{M_0} \partial_\tau \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \mathrm{d}m = \partial_\tau \int_{M_0} \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \mathrm{d}m = \partial_\tau \int_{M_\tau} \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle \rho \mathrm{d}^3 \boldsymbol{\xi}$.

** Although the Lin's constraint yields a rotational component, it is shown that the helicity of the vorticity field for a homentropic fluid in which grad$s = 0$ vanishes[19]. Such a rotational field is not general because the knotted vorticity field is excluded.

Using $\boldsymbol{\eta} = \delta\hat{\boldsymbol{\theta}} \times \boldsymbol{\xi}$, it is readily verified that this represents the conservation law of total angular momentum. In fact, using the vector identity $\langle \boldsymbol{v}, \delta\hat{\boldsymbol{\theta}} \times \boldsymbol{\xi} \rangle = \delta\hat{\boldsymbol{\theta}} \cdot (\boldsymbol{\xi} \times \boldsymbol{v})$, and noting that $\delta\hat{\boldsymbol{\theta}}$ is a constant vector, the first term is

$$\delta\hat{\boldsymbol{\theta}} \cdot \frac{\partial}{\partial \tau} \int_{M_\tau} \boldsymbol{\xi} \times \boldsymbol{v} \rho \mathrm{d}^3\boldsymbol{\xi} = \delta\hat{\boldsymbol{\theta}} \cdot \frac{\partial}{\partial \tau} \boldsymbol{L}(M_\tau) \tag{65}$$

The integral term denoted by $\boldsymbol{L}(M_\tau)$ is seen to be the total angular momntum of M_τ. Similarly, the second term is

$$\delta\hat{\boldsymbol{\theta}} \cdot \int_{\partial M_\tau} \boldsymbol{\xi} \times (p\boldsymbol{n}\mathrm{d}^2 S) = -\delta\hat{\boldsymbol{\theta}} \cdot \boldsymbol{N}(\partial M_\tau) \tag{66}$$

The surface integral over ∂M_τ is denoted by $-\boldsymbol{N}(\partial M_\tau)$, since $\boldsymbol{N}(\partial M_\tau)$ is the resultant moment of pressure force $-p\boldsymbol{n}\mathrm{d}^2 S$ acted on a surface element $\mathrm{d}^2 S$ from outside. Since $\delta\hat{\boldsymbol{\theta}}$ is an arbitrary constant vector, the Eq.(64) implies the following

$$\frac{\partial}{\partial \tau} \boldsymbol{L}(M_\tau) = \boldsymbol{N}(\partial M_\tau) \tag{67}$$

Thus, it is found that the Noether's theorem for $SO(3)$ gauge transformation leads to the conservation of total angular momentum.

6.5 Particle-permutation Symmetry and Conservation Equation

Suppose that the fluid is homentropic and has a uniform value s_0 (say) of the entropy. Then every individual particle is equivalent under adiabatic condition. Consider an adiabatic permutation of particles $\boldsymbol{a} \to \boldsymbol{a}'$ (without affecting the actual velocity field), in which the original position of a particle $\boldsymbol{a}$ in the $\boldsymbol{x}$-space is denoted by $\boldsymbol{x}_0(\boldsymbol{a})$ and the new position by $\boldsymbol{x}(\boldsymbol{a}) := \boldsymbol{x}_0(\boldsymbol{a}')$, and suppose that the mass $\mathrm{d}m = \mathrm{d}^3\boldsymbol{a}'$ at $\boldsymbol{a}'$ is replaced by the same amount of mass $\mathrm{d}m = \mathrm{d}^3\boldsymbol{a}$ at $\boldsymbol{a}$. This is expressed by*

$$\frac{\partial(a', b', c')}{\partial(a, b, c)} = 1 \tag{68}$$

Possible invariance with respect to this permutation is rephrased as an invariance with respect to a diffeomorphic transformation under the condition of volume preserving ($\mathrm{d}^3\boldsymbol{a}$ is conserved) in the mass-coordinate $\boldsymbol{a}$-space and entropy conservation.

Suppose that the permutation of particles is carried out adiabatically, hence s has the same uniform value $s(\boldsymbol{a}) = s_0$ after the permutation too, and further that the new density $\rho(\boldsymbol{a})$ of individual particles is transformed so as to have the desnity $\rho_0(\boldsymbol{a}')$ and preserve the volume $\mathrm{d}^3\boldsymbol{x}(\boldsymbol{a}) = \mathrm{d}^3\boldsymbol{x}_0(\boldsymbol{a}')$, hence the mass is conserved: $\mathrm{d}m = \rho(\boldsymbol{a})\mathrm{d}^3\boldsymbol{x}(\boldsymbol{a}) = \rho_0(\boldsymbol{a}')\mathrm{d}^3\boldsymbol{x}_0(\boldsymbol{a}')$ by the permutation.

For an infinitesimal transformation $\boldsymbol{a}' - \boldsymbol{a} = \bar{\delta}\boldsymbol{a}(\tau, \boldsymbol{a}) = (\bar{\delta}a, \bar{\delta}b, \bar{\delta}c)$, the Eq.(68) implies

$$\frac{\partial \bar{\delta}a}{\partial a} + \frac{\partial \bar{\delta}b}{\partial b} + \frac{\partial \bar{\delta}c}{\partial c} = 0$$

Following Salmon (see Ref.[19],§4), such a volume-preserving vector field $\bar{\delta}\boldsymbol{a}(\tau, \boldsymbol{a})$ is represented as

$$\bar{\delta}\boldsymbol{a} = \boldsymbol{\nabla}_a \times \bar{\delta}\boldsymbol{A}(\boldsymbol{a}) \tag{69}$$

with some vector potential $\bar{\delta}\boldsymbol{A}(\tau, \boldsymbol{a})$ where $\boldsymbol{\nabla}_a = (\partial_a, \partial_b, \partial_c)$ is the gradient operator in the $\boldsymbol{a}$-space.

Once the above permutation is carried out at an initial instant, subsequent development is governed by the equations of motion: Eqs.(61) and (42)~(44). The variation $\bar{\delta}\boldsymbol{a}$ and its associated variation field $\bar{\delta}\boldsymbol{A}$ are arbitrary, but assumed to satisfy the same boundary conditions as $\boldsymbol{\eta}$ given by Eqs.(45) and (46). Although the density, entropy and velocity field are not changed in the Eulerian sense by the above transformation, there is a change of velocity $\bar{\delta}\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{a} + \bar{\delta}\boldsymbol{a}) - \boldsymbol{v}_0(\boldsymbol{a})$ in the Lagrangian sense corresponding to an infinitesimal change $\bar{\delta}\boldsymbol{a}$. The relation between the two infinitesimal variations $\bar{\delta}\boldsymbol{a}$ and $\bar{\delta}\boldsymbol{v}$ is obtained from Eq.(44) as $\partial_t \bar{\delta}a^i + (\boldsymbol{v} \cdot \nabla)\bar{\delta}a^i + (\bar{\delta}\boldsymbol{v} \cdot \nabla)a^i = 0$. Solving for $\bar{\delta}v^j$ in the term $(\bar{\delta}\boldsymbol{v} \cdot \nabla)a^i = \bar{\delta}v^j(\partial a^i/\partial x^j)$, we obtain

$$\begin{aligned} \bar{\delta}v^k &= -\frac{\partial x^k}{\partial a^i}\, \partial_\tau \bar{\delta}a^i(t, \boldsymbol{x}) \\ \partial_\tau \bar{\delta}a^i(t, \boldsymbol{x}) &= \partial_t \bar{\delta}a^i + (\boldsymbol{v} \cdot \nabla)\bar{\delta}a^i \end{aligned} \tag{70}$$

where $(\partial x^k/\partial a^i)(\partial a^i/\partial x^j)\bar{\delta}v^j = \delta^k_j \bar{\delta}v^j = \bar{\delta}v^k$ for a map $x^k(\tau, \boldsymbol{a})$ and its inverse $a^i(t, \boldsymbol{x})$ with $\tau = t$. This is not a real change of the Eulerian velocity field in the $\boldsymbol{x}$-space. Therefore, the action with the Lagrangian (48) should not change by this permutation of indistinguishable particles, and have a permutation symmetry**.

The second integral of Eq.(48) is invariant because ρ and $\mathrm{d}^3 m = \rho \mathrm{d}^3\boldsymbol{\xi}$ are not changed, while the first integral is influenced by $\bar{\delta}\boldsymbol{v}$. Thus, the invariance of the action is represented by

$$0 = \bar{\delta} \int \Lambda_{\text{fluid}} \mathrm{d}\tau = \int \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{a}\; v^k \bar{\delta}v^k$$

* The expression (68) is called unimodular or measure-preserving transformation by Eckart[17].

** This was connected with the particle-relabeling symmetry[19]. However, this must be interpreted by a real physical symmetry of permutation of indistinguishable particles, rather than unphysical relabeling, although the relabeling leaves the desnity and entropy unchanged as well.

$$= -\int \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{a}\ v^k \frac{\partial x^k}{\partial a^i}\partial_\tau \bar{\delta} a^i$$
$$= -\int \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{a}(\boldsymbol{V}\cdot\partial_\tau \bar{\delta}\boldsymbol{a}) \tag{71}$$

where

$$\boldsymbol{V} = v^k \nabla_a x^k \tag{72}$$

Using Eq.(69) and carrying out integration by parts, we obtain

$$\bar{\delta}\int \boldsymbol{\Lambda}_{\text{fluid}}\mathrm{d}\tau = -\int \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{a}\big(\boldsymbol{V}\cdot\partial_\tau(\nabla_a\times\bar{\delta}\boldsymbol{A})\big)$$
$$= \int \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{a}\big(\partial_\tau(\nabla_a\times\boldsymbol{V})\cdot\bar{\delta}\boldsymbol{A}\big)$$

Since the variation field $\bar{\delta}\boldsymbol{A}$ is arbitrary, the action principle requires

$$\partial_\tau(\nabla_a\times\boldsymbol{V}) = 0 \tag{73}$$

This equation, discovered by Eckart (1960)[17,19], represents a conservation law in the particle-coordinate space. Its implication is considered in the next subsection.

6.6 Equation of Vorticity and Kelvin's Circulation Theorem

To see the meaning of Eq.(73) in the $\boldsymbol{x}$-space, let $\varphi(a,b,c)$ be any scalar function of particle coordinates. Using Eqs.(72) and (40) for the definition of $J=\partial(\boldsymbol{x})/\partial(\boldsymbol{a})$, we have the following identity[19]

$$(\nabla_a\times\boldsymbol{V})\cdot\nabla_a\varphi = J(\nabla\times\boldsymbol{v})\cdot\nabla\varphi \tag{74}$$

Since $\partial_\tau\nabla_a\varphi=0$, we obtain $\partial_\tau[(\nabla_a\times\boldsymbol{V})\cdot\nabla_a\varphi]=0$ from Eq.(73), equivalently we have

$$\partial_\tau[J(\nabla\times\boldsymbol{v})\cdot\nabla\varphi] = 0 \tag{75}$$

Recalling that $J\mathrm{d}^3\boldsymbol{a}=\mathrm{d}^3\boldsymbol{\xi}$ is a volume three-form and $\partial_\tau \mathrm{d}^3\boldsymbol{a}=0$, this equation is

$$\partial_\tau[(\nabla\times\boldsymbol{v})\cdot\nabla\varphi\mathrm{d}^3\boldsymbol{\xi}] = 0 \tag{76}$$

In terms of the differential forms[11], the gradient of a scalar function $\nabla\varphi$ is described by a one-form F^1, and the curl of a vector, $\boldsymbol{\omega}=\nabla\times\boldsymbol{v}$, is described by a two-form $\boldsymbol{\Omega}^2$

$$F^1 = \partial_x\varphi\mathrm{d}x + \partial_y\varphi\mathrm{d}y + \partial_z\varphi\mathrm{d}z$$
$$\boldsymbol{\Omega}^2 = \omega_x\mathrm{d}y\wedge\mathrm{d}z + \omega_y\mathrm{d}z\wedge\mathrm{d}x + \omega_z\mathrm{d}x\wedge\mathrm{d}y$$

Their exterior product is equivalent to the term inside the [ ] of Eq.(76)

$$\begin{aligned} F^1\wedge\boldsymbol{\Omega}^2 &= \nabla\varphi\cdot(\nabla\times\boldsymbol{v})\mathrm{d}^3\boldsymbol{\xi} \\ \mathrm{d}^3\boldsymbol{\xi} &= \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \end{aligned} \tag{77}$$

The derivative $\partial/\partial\tau$ is understood as the Lie derivative $\mathcal{L}_X$ with $X=\partial_t+v^i(t,\boldsymbol{x})\partial_i$ (see the footnote to §6.1). Applying the calculus of differential forms[11], the Eq.(76) is equivalent to

$$0 = \mathcal{L}_X(F^1\wedge\boldsymbol{\Omega}^2) = \mathcal{L}_X(F^1)\wedge\boldsymbol{\Omega}^2 +$$
$$F^1\wedge\mathcal{L}_X(\boldsymbol{\Omega}^2) = F^1\wedge\mathcal{L}_X(\boldsymbol{\Omega}^2)$$

since $\mathcal{L}_X(F^1)=\mathcal{L}_X\circ\mathrm{d}\varphi=\mathrm{d}\circ\mathcal{L}_X\varphi=\mathrm{d}\circ\partial_\tau\varphi=0$. Assuming $F^1\neq0$, we obtain*

$$\mathcal{L}_X(\boldsymbol{\Omega}^2)=0 \Rightarrow \partial_t\boldsymbol{\omega}+\mathrm{curl}(\boldsymbol{\omega}\times\boldsymbol{v})=0 \tag{78}$$

Thus, the vorticity equation has been derived from the conservation law Eq.(73) associated with the symmetry of particle permutation. Using the vector identity, $\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v})=(\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}+(\nabla\cdot\boldsymbol{v})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}$ (since $\nabla\cdot\boldsymbol{\omega}=0$) together with the continuity Eq.(42), this is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\boldsymbol{v} \tag{79}$$

In addition, the law (73) leads to the Kelvin's circulation theorem[19]. Consider a closed loop C_a in $\boldsymbol{a}$-space and denote its line element by $\mathrm{d}\boldsymbol{a}$. Using Eq.(72), we have

$$\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{a} = \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\xi} \tag{80}$$

where $\mathrm{d}\boldsymbol{\xi}$ is a corresponding line element in physical $\boldsymbol{\xi}$-space (equivalently $\boldsymbol{x}$-space). From Eq.(73), we obtain the following integration law

$$\partial_\tau\oint_{C_a}\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{a} = \partial_\tau\int_{S_a}(\nabla_a\times\boldsymbol{V})\cdot\mathrm{d}\boldsymbol{S}_a$$
$$= \int_{S_a}\partial_\tau(\nabla_a\times\boldsymbol{V})\cdot\mathrm{d}\boldsymbol{S}_a = 0 \tag{81}$$

where $\boldsymbol{S}_a$ and $\mathrm{d}\boldsymbol{S}_a$ are an open surface bounded by the loop C_a and its surface element in $\boldsymbol{a}$-space, respectively. Thus, using Eq.(80), we obtain the Kelvin's circulation theorem

$$\partial_\tau\oint_{C_a(\xi)}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\xi} = 0 \tag{82}$$

where $C_a(\boldsymbol{\xi})$ is a closed material loop in $\boldsymbol{\xi}$-space ($\boldsymbol{x}$-space) corresponding to C_a.

* $\mathcal{L}_{\partial_t}(\boldsymbol{\Omega}^2)$ yields $\partial_t\boldsymbol{\omega}$, whereas $\mathcal{L}_{v^i\partial_i}(\boldsymbol{\Omega}^2)$ yields $\mathrm{curl}(\boldsymbol{\omega}\times\boldsymbol{v})+\mathrm{div}(\boldsymbol{\omega})\boldsymbol{v}=\mathrm{curl}(\boldsymbol{\omega}\times\boldsymbol{v})$.

Bretherton[18] considered the invariance of the action integral under a reshuffling of indistinguishable particles which leaves the fields of velocity, density and entropy unaltered, and derived the Kelvin's circulation theorem directly. The present derivation is advantageous in the sense that a local form Eq.(73) is derived.

The description in this section aims to unify the contributions of Refs.[17∼19] from the point of view of adiabatic particle permutation symmetry.

7 SUMMARY AND CONCLUSIONS

Following the gauge principle in the quantum field theory, the present paper tried to find a scenario in fluid flow which has a formal equivalence with the gauge theory. The free-field Lagrangian defined initially satisfies global gauge invariance as well as Galilei invariance. However, the equation derived from the variational principle does not satisfy local gauge invariance. This is because local gauge transformation of velocity $\boldsymbol{v}(x)$ requires the vorticity in the velocity field, whereas the action principle applied to the initial Lagrangian results in the equations for potential flows, i.e. irrotational flows. In complying with the local gauge invariance, a gauge-covariant derivative is defined in terms of a gauge field. The Galilei invariance requires that the gauge field should coincide with the vorticity. As a result, the covariant derivative of velocity is found to be given by the material time derivative of velocity.

Using the gauge-covariant derivative, a new variational formulation is attempted successfully by means of isentropic material variations, and the Euler's equation of motion is derived for rotational isentropic flows from the Hamilton's principle. This formulation is considered to unify traditional two approaches. Namely, the Eulerian approach is used to derive an irrotational free-field which does not necessarily specify the motion $\boldsymbol{\xi}(t)$ of individual material particles, while the Lagrangian approach is used for full material variation including rotational velocity field to satisfy local gauge invariance.

There are some byproducts from the present formulation. The global $SO(3)$ gauge invariance infers a Noether's conservation law, which is found to be the conservation of total angular momentum. In addition, the Lagrangian has a symmetry with respect to particle permutation, which leads to a local law of vorticity conservation, i.e. the vorticity equation as well as the Kelvin's circulation theorem. Thus, it is found that the well-known equations are related to certain symmetries of the fluid system.

The present gauge theory provides a theoretical ground for physical analogy between the aeroacoustic phenomena associated with vortices[3,4,8,9] and the electron and electromagnetic-field interactions. In particular, it is considered that the Aharonov-Bohm effect in quantum mechanics[7] has a direct analogy with the scattering of sound waves or shallow water waves by the interaction with vortices[5,6].

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Appendix

Additional Description to the Gauge Theory in Section 2

The replacement of ∂_μ with $\nabla_\mu = \partial_\mu - \mathrm{i}q\boldsymbol{A}_\mu(x)$ defined by Eq.(5) in the quantum electrodynamics of Section 2 (1) introduces in the Lagrangian (1) an interaction term $-\boldsymbol{A}_\mu \boldsymbol{J}^\mu$ between the gauge field $\boldsymbol{A}_\mu$ and the electromagnetic current density (matter field) $\boldsymbol{J}^\mu = -q\bar{\boldsymbol{\psi}}\boldsymbol{\gamma}^\mu\boldsymbol{\psi}$.

To arrive at the complete Lagrangian, it remains to add an electromagnetic field term $-\frac{1}{16\pi}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$ to the Lagrangian, where $\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu$, and $\boldsymbol{F}^{\mu\nu} = \boldsymbol{g}^{\mu\alpha}\boldsymbol{F}_{\alpha\beta}\boldsymbol{g}^{\beta\nu}$ with the metric tensor $\boldsymbol{g}^{\alpha\beta}$ =diag$(-1,1,1,1)$. Assembling all the pieces, we have the Lagrangian for quantum electrodynamics

$$\boldsymbol{\Lambda}_{\mathrm{qed}} = \boldsymbol{\Lambda}_{\mathrm{free}} - \boldsymbol{A}_\mu \boldsymbol{J}^\mu - \frac{1}{16\pi}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} \tag{A1}$$

Thus, variation with respect to $\boldsymbol{A}_\mu$ yield the equations for the gauge field $\boldsymbol{A}_\mu$, i.e. Maxwell's equations in electromagnetism, whereas variation of $\boldsymbol{\Lambda}_{\mathrm{qed}}$ with respect to $\boldsymbol{\psi}$ yields the equation of quantum electrodynamics, i.e. Dirac equation with electromagnetic field.

Using the notations $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla\phi - c^{-1}\partial_t \boldsymbol{A}$ of the magnetic three-vector $\boldsymbol{B}$ and electric three-vector $\boldsymbol{E}$, we have

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

In the Yang-Mills's system of Section 2 (2), the gauge-covariant derivative is defined by $\nabla_\mu = \partial_\mu - \mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu(x)$. In addition, the following three gauge fields (colors) are defined: $\boldsymbol{A}^k = (A_0^k, A_1^k, A_2^k, A_3^k)$ with $k = 1,2,3$. The new fields $\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3$ are the Yang-Mills gauge fields. The connection $\mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu$ leads to the interaction term, that couples the gauge field with isospin currrent, corresponding to the middle term of Eq.(A1). Fianlly, a gauge field term (called a kinetic term, corresponding to the third term of Eq.(A1) should be added to complete the Yang-Mills action functional.